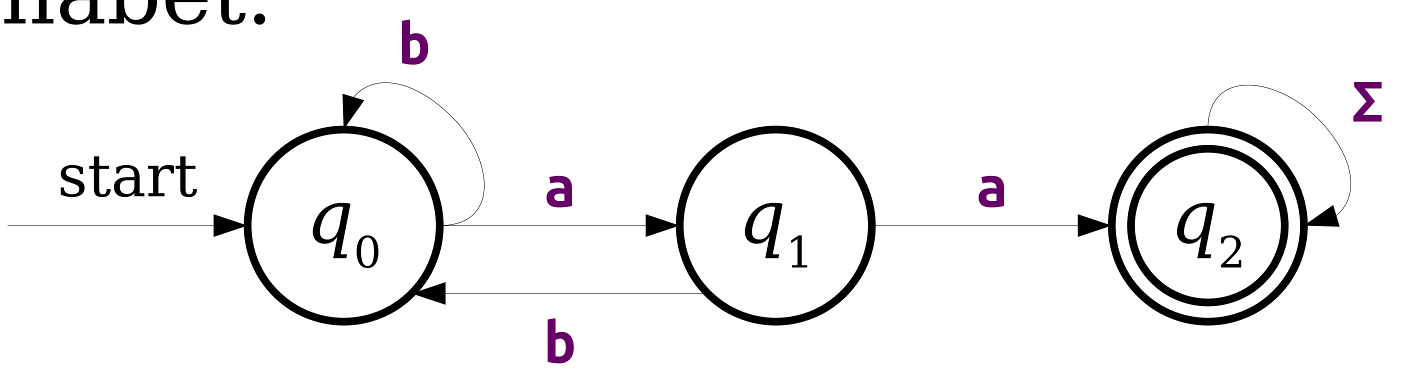
Finite Automata

Part Three

Recap from Last TIme

DFAs

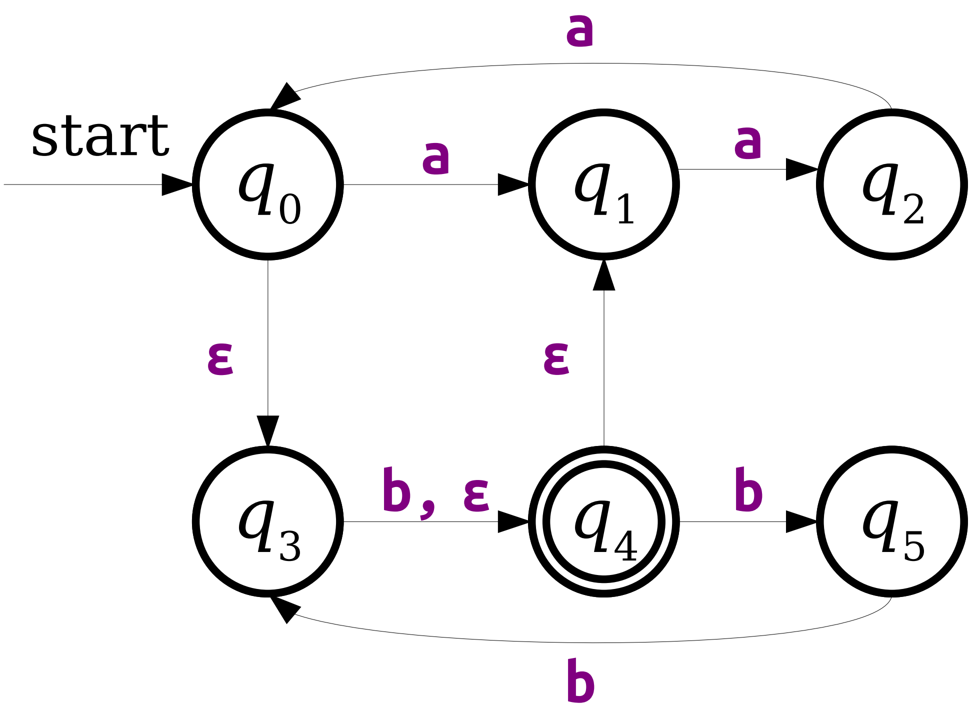
- A ***DFA*** is a
  - ***D***eterministic
  - ***F***inite
  - ***A***utomaton
- Every state must have exactly one transition defined for each symbol in the alphabet.



If D is a DFA, the ***language of D***, denoted $\mathscr{L}(D)$, is $\{ w \in \Sigma^* \mid D \text{ accepts } w \}$.

NFAs

- An ***NFA*** is a
  - ***N***ondeterministic
  - ***F***inite
  - ***A***utomaton
- NFAs have no restrictions on how many transitions are allowed per state.
- They can also use ε-transitions.
- An NFA accepts a string *w* if there is some sequence of choices that leads to an accepting state.



Massive Parallelism

- An NFA can be thought of as a DFA that can be in many states at once.
- At each point in time, when the NFA needs to follow a transition, it tries all the options at the same time.
- The NFA accepts if *any* of the states that are active at the end are accepting states. It rejects otherwise.

New Stuff!

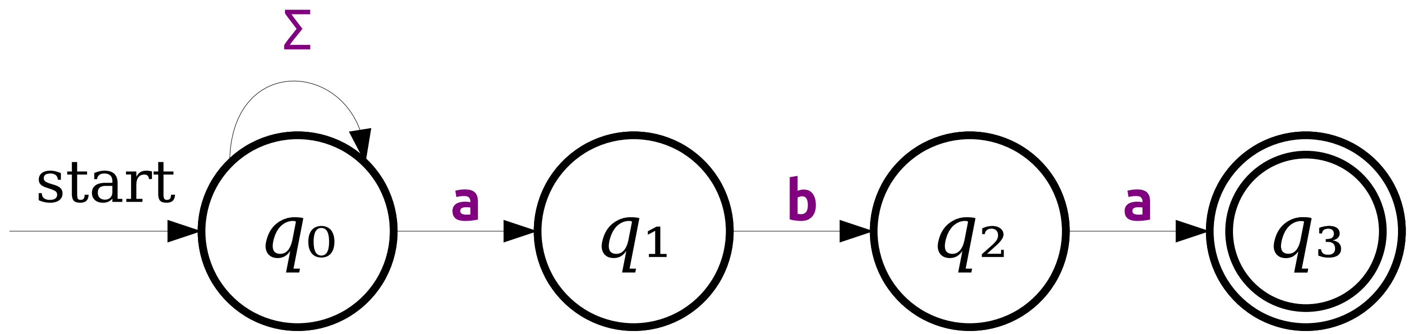
Just how powerful *are* NFAs?

NFAs and DFAs

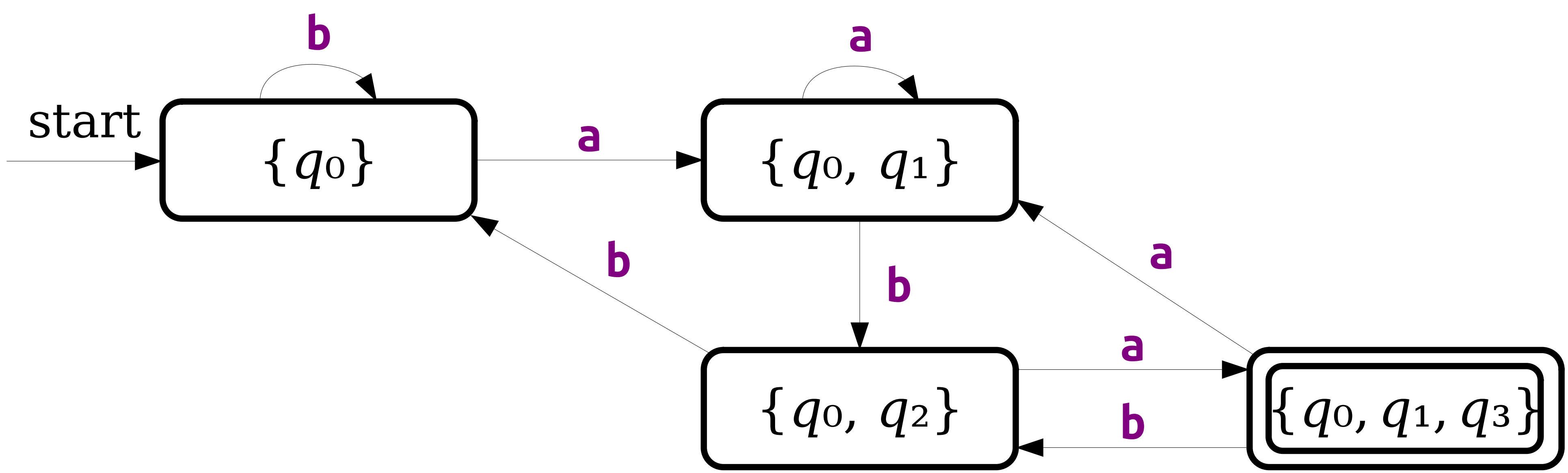
- Any language that can be accepted by a DFA can be accepted by an NFA.
- Why?
  - Every DFA essentially already *is* an NFA!
- ***Question:*** Can any language accepted by an NFA also be accepted by a DFA?
- Surprisingly, the answer is ***yes***!

Thought Experiment:

How would you simulate an NFA in software?



	a	b
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$
$\{q_0, q_2\}$	$\{q_0, q_1, q_3\}$	$\{q_0\}$
$*\{q_0, q_1, q_3\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$



The Subset Construction

- This procedure for turning an NFA for a language L into a DFA for a language L is called the ***subset construction***.
  - It's sometimes called the ***powerset construction***; it's different names for the same thing!
- Intuitively:
  - Each state in the DFA corresponds to a set of states from the NFA.
  - Each transition in the DFA corresponds to what transitions would be taken in the NFA when using the massive parallel intuition.
  - The accepting states in the DFA correspond to which sets of states would be considered accepting in the NFA when using the massive parallel intuition.
- There's an online ***Guide to the Subset Construction*** with a more elaborate example involving ε-transitions and cases where the NFA dies; check that for more details.

The Subset Construction

- In converting an NFA to a DFA, the DFA's states correspond to sets of NFA states.
- ***Useful fact:*** $|\wp(S)| = 2^{|S|}$ for any finite set S.
- In the worst-case, the construction can result in a DFA that is *exponentially larger* than the original NFA.
- ***Question to ponder:*** Can you find a family of languages that have NFAs of size n, but no DFAs of size less than 2^n?

Time-Out for Announcements!

Problem Set Six

- Problem Set Five was due at 1:00PM today.
  - You can use a late day to extend the deadline to 1:00PM Saturday.
- Problem Set Six goes out today. It’s due next Friday at 1:00PM.
  - Design DFAs and NFAs for a range of problems!
  - Explore formal language theory!
  - See some clever applications!

Back to CS103!

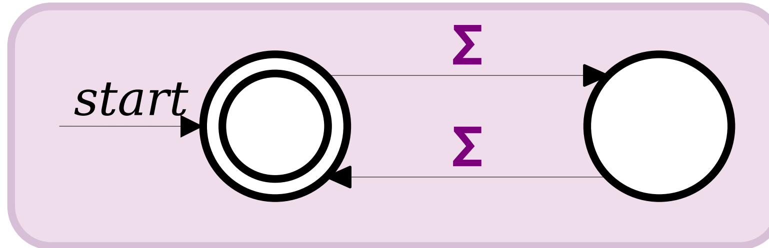
The Regular Languages

Regular Languages

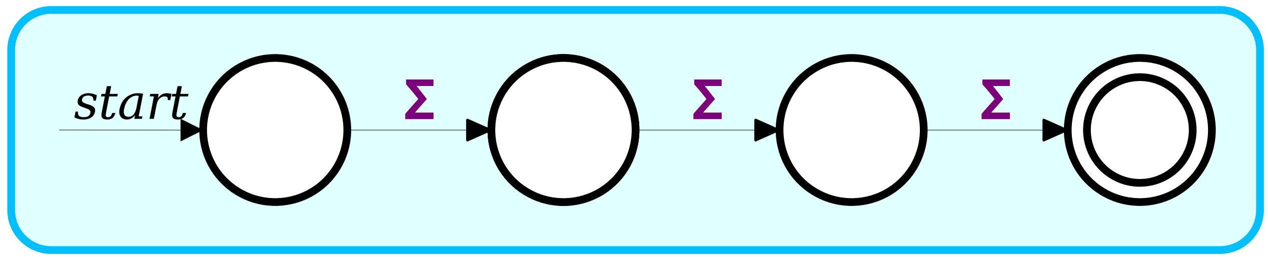
- Let $L \subseteq \Sigma^*$ be a language.
- We say that L is a ***regular language*** if there is a DFA D where $\mathscr{L}(D) = L$.
- Equivalently, L is a regular language if there is an NFA N where $\mathscr{L}(N) = L$.
- Key questions:
  - What do the regular languages "feel" like?
  - What properties do they have?
  - What languages *aren't* regular?

Closure Under Union

- If L_1 and L_2 are languages over the alphabet Σ, the language $L_1 \cup L_2$ is the language of all strings in at least one of the two languages.
- If L_1 and L_2 are regular languages, is $L_1 \cup L_2$?



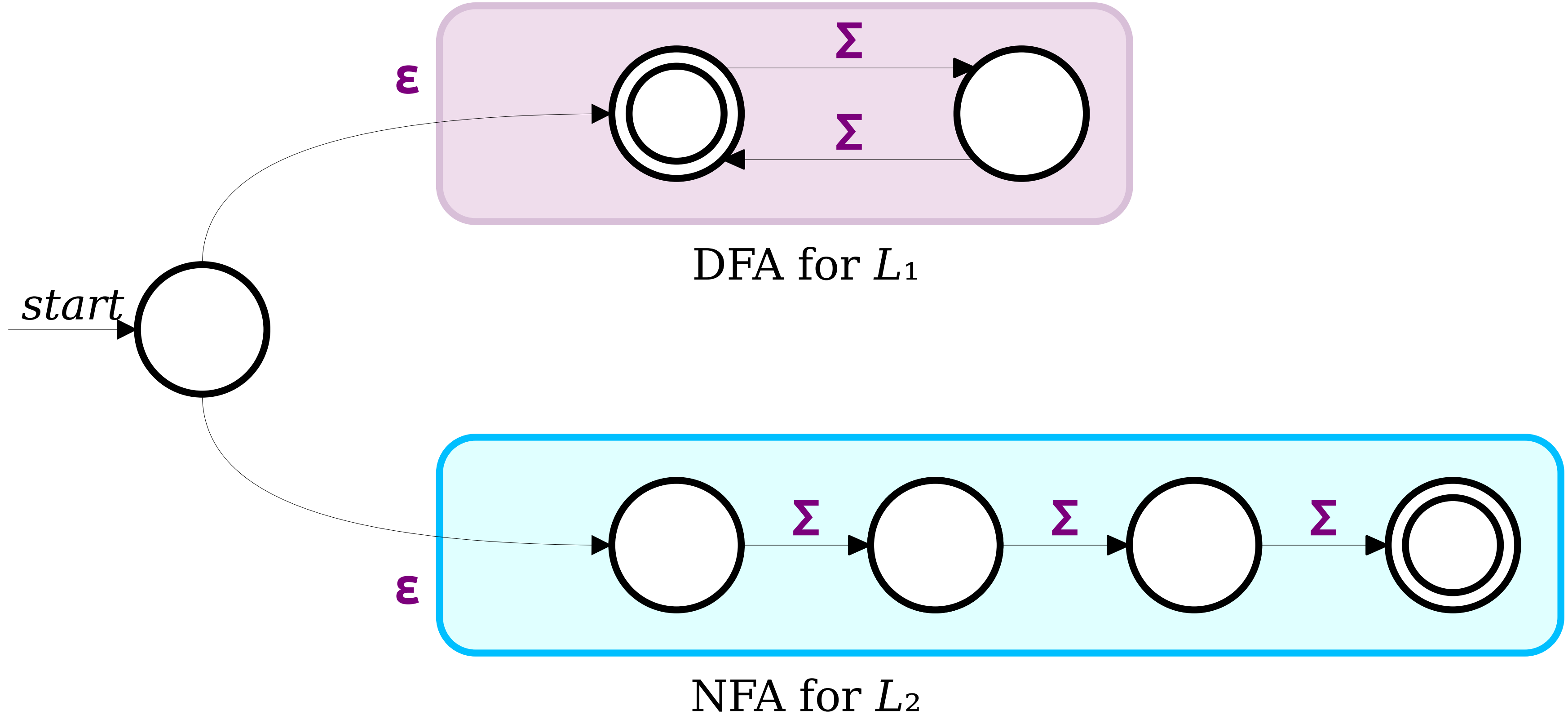
DFA for L_1



NFA for L_2

L_1 = { w ∈ {**a**, **b**}* | w has even length }
L_2 = { w ∈ {**a**, **b**}* | w has length exactly three }

Construct an NFA for $L_1 \cup L_2$.



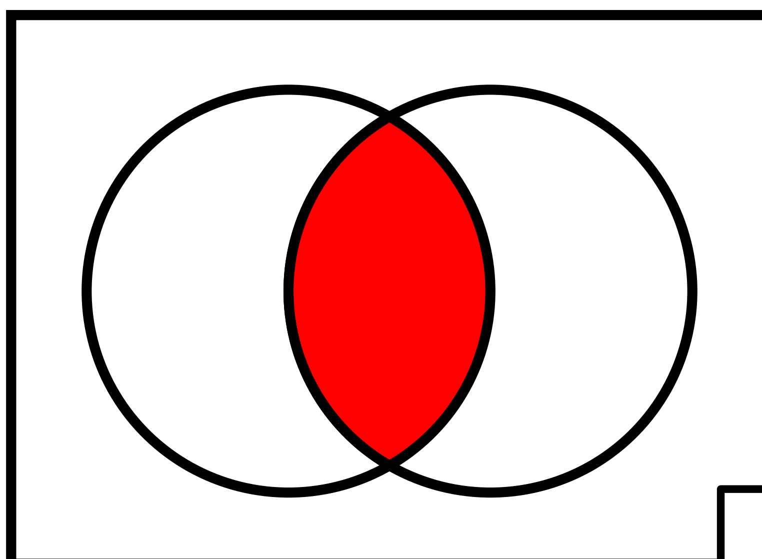
$L_1 = \{\ w \in \{\mathbf{a}, \mathbf{b}\}^* \mid w \text{ has even length }\}$

$L_2 = \{\ w \in \{\mathbf{a}, \mathbf{b}\}^* \mid w \text{ has length exactly three }\}$

Construct an NFA for $L_1 \cup L_2$.

Closure Under Intersection

- If L_1 and L_2 are languages over Σ, then $L_1 \cap L_2$ is the language of strings in both L_1 and L_2.
- Question: If L_1 and L_2 are regular, is $L_1 \cap L_2$ regular as well?



$$\overline{\overline{L_1} \cup \overline{L_2}}$$

Hey, it's De Morgan's laws!

Concatenation

String Concatenation

- If $w \in \Sigma^*$ and $x \in \Sigma^*$, the ***concatenation*** of w and x, denoted ***wx***, is the string formed by tacking all the characters of x onto the end of w.
- Example: if w = **quo** and x = **kka**, the concatenation wx = **quokka**.
- This is analogous to the + operator for strings in many programming languages.
- Some facts about concatenation:
  - The empty string ε is the ***identity element*** for concatenation:

$$w\varepsilon = \varepsilon w = w$$

  - Concatenation is ***associative***:

$$wxy = w(xy) = (wx)y$$

Concatenation

- The ***concatenation*** of two languages L_1 and L_2 over the alphabet Σ is the language

$$L_1L_2 = \{ x \mid \exists w_1 \in L_1.\ \exists w_2 \in L_2.\ x = w_1w_2 \}$$

- Let L_1 = { **ab**, **ba** } and L_2 = { **aa**, **bb** }. What is L_1L_2?

Answer at
https://pollev.com/cs103aut23

Concatenation Example

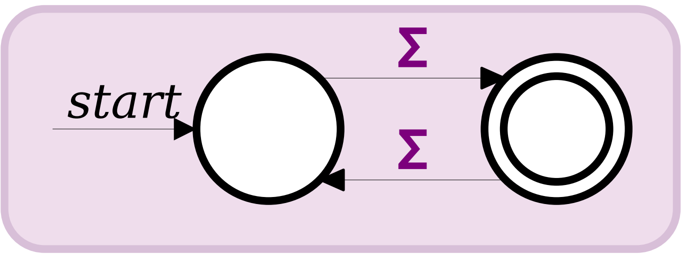
- Let Σ = { **a**, **b**, …, **z**, **A**, **B**, …, **Z** } and consider these languages over Σ:
  - ***Noun*** = { **Puppy**, **Rainbow**, **Whale**, … }
  - ***Verb*** = { **Hugs**, **Juggles**, **Loves**, … }
  - ***The*** = { **The** }
- The language ***TheNounVerbTheNoun*** is
  - { **ThePuppyHugsTheWhale**, **TheWhaleLovesTheRainbow**, **TheRainbowJugglesTheRainbow**, … }

Concatenation

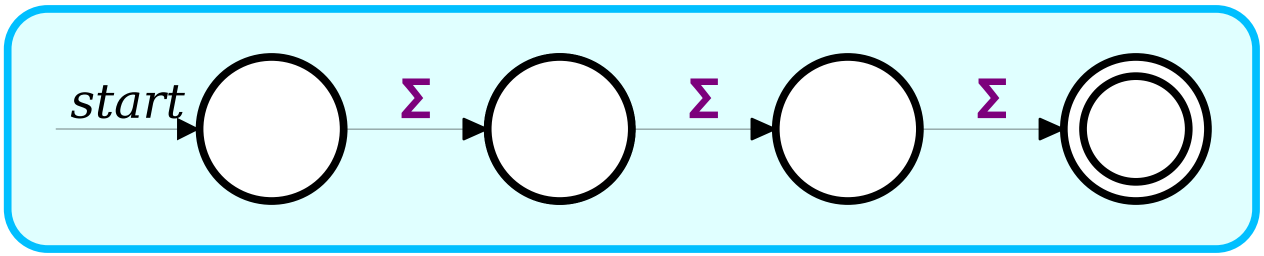
- The ***concatenation*** of two languages L_1 and L_2 over the alphabet Σ is the language

$$\boldsymbol{L_1L_2 = \{\ x \mid \exists w_1 \in L_1.\ \exists w_2 \in L_2.\ x = w_1w_2\ \}}$$

- Two views of L_1L_2:
  - The set of all strings that can be made by concatenating a string in L_1 with a string in L_2.
  - The set of strings that can be split into two pieces: a piece from L_1 and a piece from L_2.



DFA for L_1

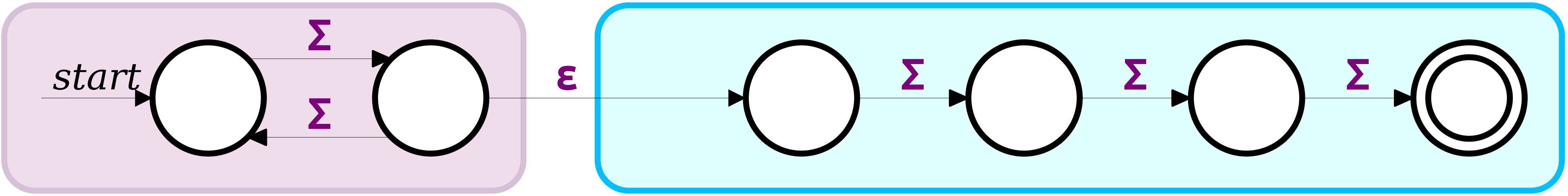


NFA for L_2

$L_1 = \{ w \in \{\texttt{a}, \texttt{b}\}^* \mid w \text{ has } \textit{odd} \text{ length } \}$

$L_2 = \{ w \in \{\texttt{a}, \texttt{b}\}^* \mid w \text{ has length exactly three } \}$

Construct an NFA for L_1L_2.



DFA for L_1 NFA for L_2

$L_1 = \{ w \in \{\mathtt{a}, \mathtt{b}\}^* \mid w \text{ has } \textit{odd} \text{ length } \}$
$L_2 = \{ w \in \{\mathtt{a}, \mathtt{b}\}^* \mid w \text{ has length exactly three } \}$

Construct an NFA for L_1L_2.

The Kleene Star

Lots and Lots of Concatenation

- Consider the language L = { aa, b }
- LL is the set of strings formed by concatenating pairs of strings in L.

{ aaaa, aab, baa, bb }

- LLL is the set of strings formed by concatenating triples of strings in L.

{ aaaaaa, aaaab, aabaa, aabb, baaaa, baab, bbaa, bbb}

- $LLLL$ is the set of strings formed by concatenating quadruples of strings in L.

{ aaaaaaaa, aaaaaab, aaaabaa, aaaabb, aabaaaa, aabaab, aabbaa, aabbb, baaaaaa, baaaab, baabaa, baabb, bbaaaa, bbaab, bbbaa, bbbb}

Language Exponentiation

- We can define what it means to "exponentiate" a language as follows:
- $L^0 = \{\varepsilon\}$
  - Intuition: The only string you can form by gluing no strings together is the empty string.
  - Notice that $\{\varepsilon\} \neq \emptyset$. Can you explain why?
- $L^{n+1} = LL^n$
  - Idea: Concatenating (n+1) strings together works by concatenating n strings, then concatenating one more.
- ***Question to ponder:*** Why define $L^0 = \{\varepsilon\}$?
- ***Question to ponder:*** What is $\emptyset^0$?

The Kleene Star

The Kleene Closure

- An important operation on languages is the ***Kleene closure***, or ***Kleene star***, which is defined as

$$L^* = \{ w \in \Sigma^* \mid \exists n \in \mathbb{N}.\ w \in L^n \}$$

- Mathematically:

$$w \in L^* \quad \leftrightarrow \quad \exists n \in \mathbb{N}.\ w \in L^n$$

- Intuitively, L^* is the language all possible ways of concatenating zero or more strings in L together, possibly with repetition.
- ***Question to ponder:*** What is Ø*?

The Kleene Closure

If L = { **a**, **bb** }, then L^* = {

ε,

a, **bb**,

aa, **abb**, **bba**, **bbbb**,

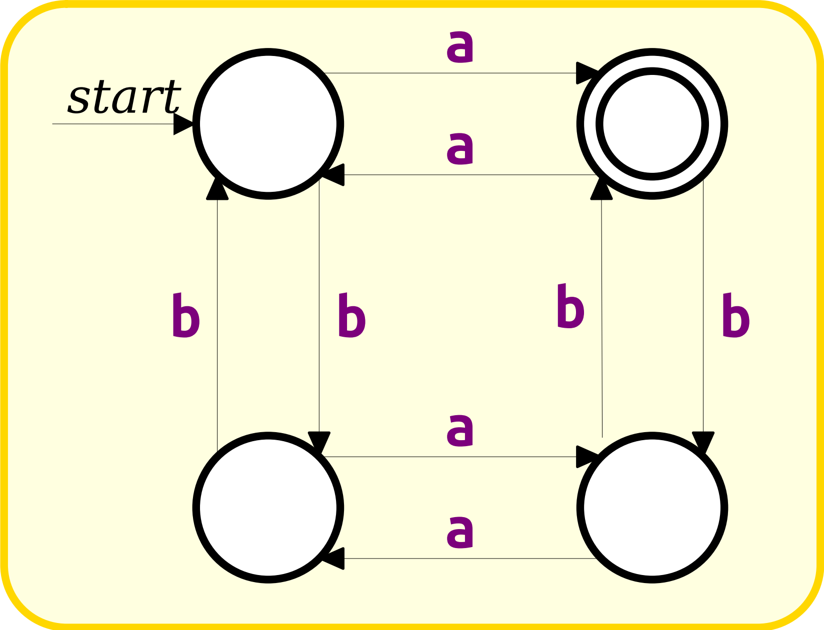
aaa, **aabb**, **abba**, **abbbb**, **bbaa**, **bbabb**, **bbbba**, **bbbbbb**,

…

}

Think of L* as the set of strings you can make if you have a collection of stamps – one for each string in L – and you form every possible string that can be made from those stamps.

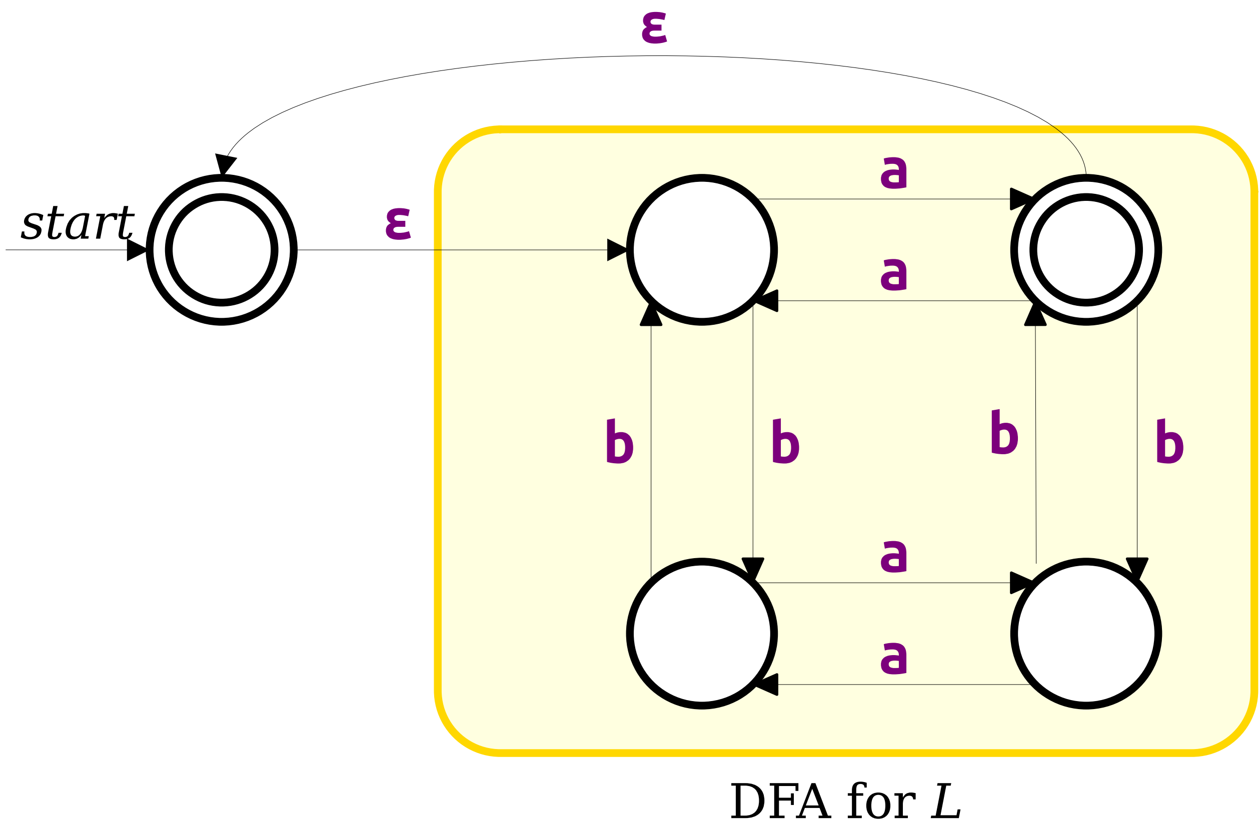
Idea: Can we convert an NFA for a language L to an NFA for language L^*?



DFA for L

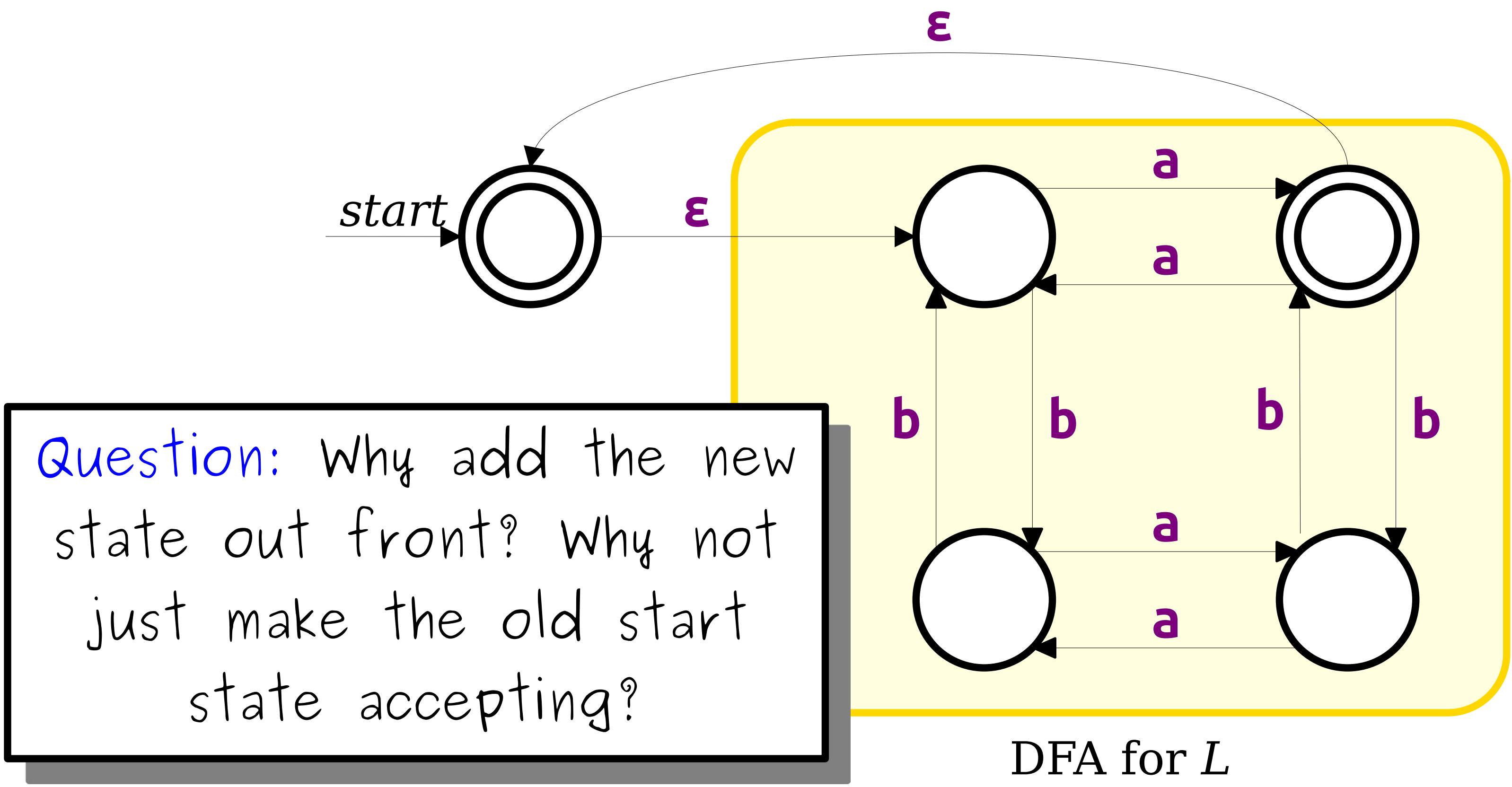
L = { w ∈ {a, b}* | w has an odd number of a's and
an even number of b's }

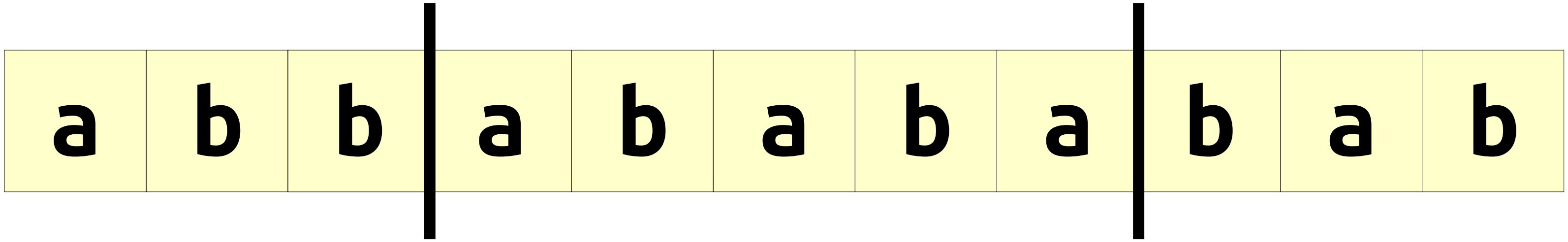
Construct an NFA for L^*.



L = { w ∈ {**a**, **b**}* | w has an odd number of **a**'s and an even number of **b**'s }

Construct an NFA for L*.





L = { $w \in$ {a, b}* | w has an odd number of a's and an even number of b's }

Construct an NFA for L^*.

Closure Properties

- ***Theorem:*** If L_1 and L_2 are regular languages over an alphabet Σ, then so are the following languages:
  - $L_1 \cup L_2$
  - $L_1 \cap L_2$
  - $L_1 L_2$
  - L_1^*
- These properties are called ***closure properties of the regular languages***.

Next Time

- ***Regular Expressions***
  - Building languages from the ground up!
- ***Thompson's Algorithm***
  - A UNIX Programmer in Theoryland.
- ***Kleene's Theorem***
  - From machines to programs!